LECTURE NOTES 2-2: THE LIMIT OF A FUNCTION

Things to Know:

- The intuitive definitions of a *limit* and a *one-sided limit*.
- How to find a (one-sided) limit using a calculator or the graph of the function, including infinite limits.
- How to find limits for piecewise-defined functions.
- How to distinguish between the various ways a limit may *not* exist.
- Understand how using a calculator can give an incorrect answer when evaluating a limit.

Intuitive Idea and Introductory Examples

we want to formalize the closeness arguement.

(Note that this is motivated by our discussion of tangent lines and instantaneous velocity.)

Say: "the limit of $f(x)$, as x approaches a is L"

Write: $\lim_{x \to a} f(x) = L$

It means: as x gets closer and closer to a, f(x) can be made arbitrarily close to L.

EXAMPLE 1: Use calculation to guess $\lim_{x \to 2} \dfrac{x-2}{x^2-x-2}$.

Pick values close to 2, plug them in and see what happens.

Q: why is simply plugging in 2 not going to work?

A: You get 0/0, undefined.

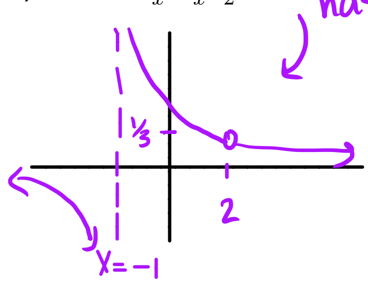
Let $f(x) = \dfrac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	0.34483	0.33445	0.33344	?	0.33322	0.33223	0.32258

guess: $\lim_{x \to 2} \dfrac{x-2}{x^2-x-2} = \boxed{1/3}$

What does the table above tell you about the *graph* of $y = \frac{x-2}{x^2-x-2}$?

as x gets close to 2, f(x) (i.e. y) gets close to 1/3



has a hole at (2, 1/3)

$\boxed{\text{EXAMPLE 2:}}$ [Why do all the calculation? Just pick a number really close to "a," right???!!]

$$\text{Use calculation to guess } \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}.$$

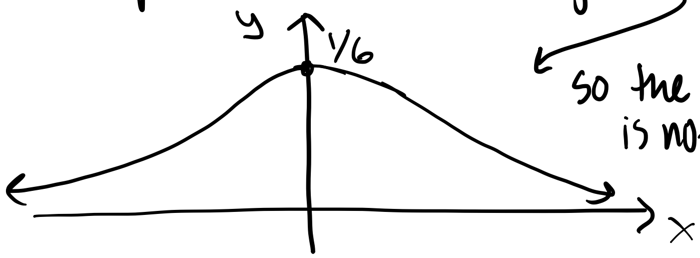
Let's just pick numbers super-close to $a = 0$, say ± 0.000001 :

t	-0.000001	0	0.000001
$f(t)$	0.1665335	DNE	0.1665335

Hint: Always be skeptical! Why can't this be right and what went wrong?

↑ Beth got this using a computer, but some calculators give 0 for both of these.

If we let $f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$ you see the graph of $f(t)$ is (roughly)



so the limit is not zero

The numerator gets <u>so</u> small the calculator rounds it to <u>zero</u>.

$\boxed{\text{EXAMPLE 3:}}$ [Sample points may not illustrate the big picture. Theory will be useful.]

$$\text{Use calculation to guess } \lim_{\theta\to 0} \sin\left(\frac{\pi}{\theta}\right).$$

Let $f(\theta) = \sin(\pi/\theta)$

	$-1/10$	$-1/1000$	$-1/10000$		$1/10000$	$1/1000$	$1/10$
θ	-0.1	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(\theta)$	①	②	③		④	⑤	⑥

Do you believe your answer?

① $\sin\left(\frac{\pi}{(-1/10)}\right) = \sin(-10\pi) = 0$

② $\sin\left(\pi/(-1/1000)\right) = \sin(-1000\pi) = 0$

③ $\sin\left(\pi/(-1/10000)\right) = \sin(-10{,}000\pi) = 0$

④ $\sin\left(\pi/(1/10000)\right) = \sin(10{,}000\pi) = 0$

⑤ $\sin\left(\pi/(1/1000)\right) = \sin(1000\pi) = 0$

⑥ $\sin\left(\pi/(1/10)\right) = \sin(10\pi) = 0.$

guess $\lim_{\theta\to 0} \sin\left(\frac{\pi}{\theta}\right) = 0$, <u>BUT</u> the graph:

is crazy!

This limit Does Not Exist!

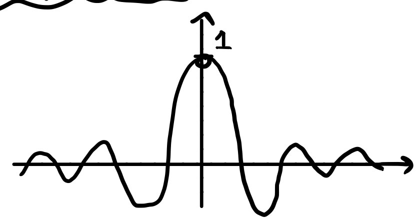
1. For each problem below, fill out the chart of values, then use the values to *guess* the value of the limit. Finally rate your confidence level on a 0 to 3 scale where (0 = I'm sure this is wrong) and (3 = I'm sure this is right.)

(a) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \boxed{1}$ confidence? _______

θ	-1	-0.5	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.5	1
$f(\theta)$	0.8415	0.9589	0.9983	0.9999	0.9999		0.9999	0.9999	0.9983	0.9589	0.8415

guess this limit goes to 1 :

picture :



(b) $\lim_{x \to 2} f(x) = \boxed{\text{DNE}}$ where $\begin{cases} |x-1| & x \leq 2 \\ x+1 & x > 2 \end{cases}$ confidence? _______

x	1	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5	3
$f(x)$	0	0.5	0.9	0.99	0.999		3.001	3.01	3.1	3.5	4

x is less than 2, use $f(x) = |x-1|$

x is greater than 2, use $x+1$

as x increases to 2, $f(x) \to 1$

as x decreases to 2, $f(x) \to 3$

these don't match, so the limit does not exist.

(c) $\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \boxed{2}$ confidence? _______

x	-0.5	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1	0.5
$f(x)$	1.264	1.813	1.98	1.998	1.9998	?	2.0002	2.002	2.02	2.214	3.44

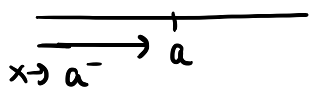
increasing to 2 → ← decreasing to 2

DEFINITIONS:

Say: "the limit as x approaches a *on the left* is L"; Write: $\boxed{\lim_{x\to a^-} f(x) = L}$

It means $x \to a^-$ means x is LESS than a, and thus is on the left side of a

diagram:



(here $x < a$) always

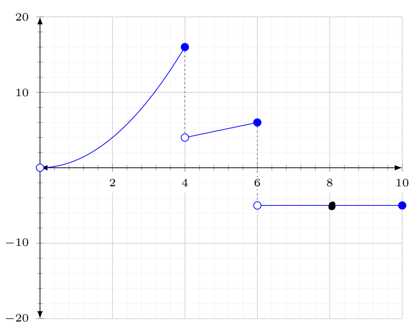
Say: "the limit as x approaches a *on the right* is L"; Write: $\boxed{\lim_{x\to a^+} f(x) = L}$

It means $x \to a^+$ means x is greater than a, and is thus on the right side of a.

(here $x > a$) always

Practice Problems

2. The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x\to 4^-} f(x) =$ 16 (4 from the left)

(b) $\lim_{x\to 4^+} f(x) =$ 4 (4 from the right)

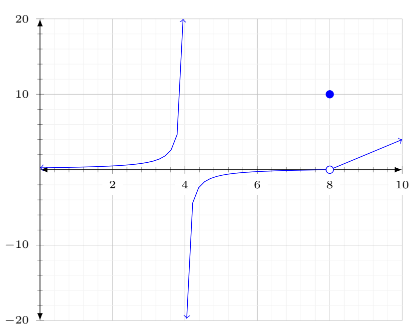
(c) $\lim_{x\to 4} f(x) =$ DNE ← WebAssign is case-sensitive.

(d) $f(4) =$ 16

(e) $\lim_{x\to 8} f(x) =$ −5

(f) $f(8) =$ −5

3. The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x\to 4^-} f(x) =$ ∞ ← gets hugely big

(b) $\lim_{x\to 4^+} f(x) =$ $-\infty$ ← gets hugely negative big

(c) $\lim_{x\to 4} f(x) =$ DNE

(d) $f(4) =$ DNE / undefined

(e) $\lim_{x\to 8} f(x) =$ 0

(f) $f(8) =$ 10

Write the equation of any vertical asymptote:

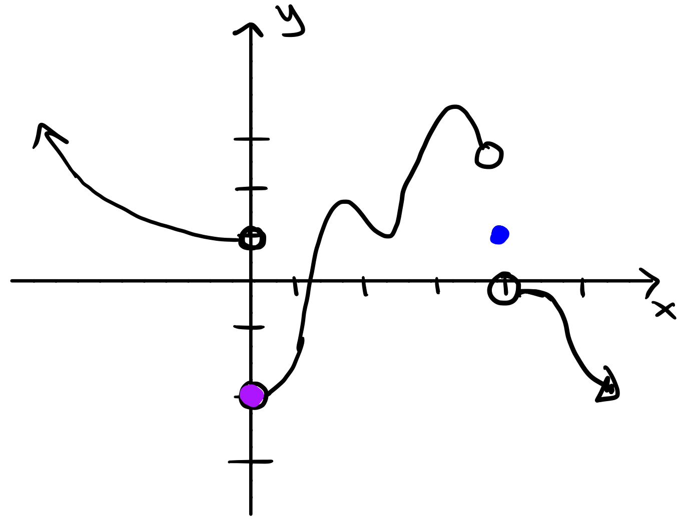
$\boxed{x = 4}$

↑ must do this! Saying "4" is not enough!

4. Sketch the graph of an function that satisfies *all* of the given conditions. Compare your answer with that of your neighbor.

$\lim_{x\to 0^-} f(x) = 1$ $\quad\lim_{x\to 0^+} f(x) = -2$ $\quad\lim_{x\to 4^-} f(x) = 3$

$\lim_{x\to 4^+} f(x) = 0$ $\quad f(0) = -2$ $\quad f(4) = 1$



There are many correct graphs for this problem →

note: constant/small pos.# → ∞

constant/small neg# → $-\infty$

5. Determine the limit. Explain your answer.

(a) $\lim_{x\to 5^+} \frac{2+x}{x-5} = \boxed{\infty}$

① numerator: as $x \to 5^+$ the numerator approaches 7

② as $x \to 5^+$, $x > 5$, so $x-5$ is positive, thus the denominator is a small, positive number

Thus the limit goes to $\boxed{\infty}$

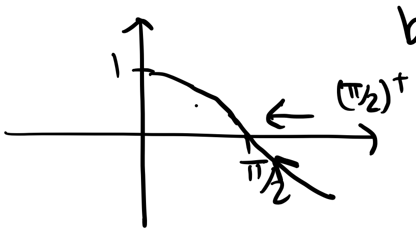
(b) $\lim_{x\to 5^+} \frac{2+x}{5-x} = \boxed{-\infty}$

① numerator approaches 7

② denominator, $x \to 5^+$ means $x > 5$, so $5-x$ is going to zero but is negative.

Thus the limit goes to $\boxed{-\infty}$

(c) $\lim_{x\to(\pi/2)^+} \frac{\sec x}{x} = \lim_{x\to \pi/2^+} \frac{1}{x\cos x}$ as $x \to \pi/2^+$ $\cos x \to 0$ but is negative Thus $1/x\cos x \to \boxed{-\infty}$



4. Sketch the graph of an function that satisfies *all* of the given conditions. Compare your answer with that of your neighbor.

$\lim_{x \to 0^-} f(x) = 1$ $\quad$ $\lim_{x \to 0^+} f(x) = -2$ $\quad$ $\lim_{x \to 4^-} f(x) = 3$

$\lim_{x \to 4^+} f(x) = 0$ $\quad$ $f(0) = -2$ $\quad$ $f(4) = 1$

5. Determine the limit. Explain your answer.

(a) $\displaystyle\lim_{x \to 5^+} \frac{2+x}{x-5}$

(b) $\displaystyle\lim_{x \to 5^+} \frac{2+x}{5-x}$

(c) $\displaystyle\lim_{x \to (\pi/2)^+} \frac{\sec x}{x}$